

$W^{1,p}$ versus C^1 : The nonsmooth case involving critical growth

Yunru Bai

*Guangxi Colleges and Universities Key Laboratory of Complex System
Optimization and Big Data Processing
Yulin Normal University, Yulin 537000, China*
and

*Jagiellonian University in Krakow
Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6, Krakow 30-348, Poland
yunrubai@163.com*

Leszek Gasiński

*Pedagogical University of Cracow
Department of Mathematics
Podchorążych 2, 30-084 Cracow, Poland
leszek.gasinski@up.krakow.pl*

Patrick Winkert

*Technische Universität Berlin, Institut für Mathematik
Strasse des 17. Juni 136, 10623 Berlin, Germany
winkert@math.tu-berlin.de*

Shengda Zeng*

*Guangxi Colleges and Universities Key Laboratory of Complex System
Optimization and Big Data Processing
Yulin Normal University, Yulin 537000, China*
and

*Jagiellonian University in Krakow
Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6
Krakow 30-348, Poland
zengshengda@163.com*

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*Corresponding author.

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In this paper, we study a class of generalized and not necessarily differentiable functionals of the form

$$J(u) = \int_{\Omega} G(x, \nabla u) dx - \int_{\Omega} j_1(x, u) dx - \int_{\partial\Omega} j_2(x, u) d\sigma$$

with functions $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ that are only locally Lipschitz in the second argument and involving critical growth for the elements of their generalized gradients $\partial j_k(x, \cdot)$, $k = 1, 2$ even on the boundary $\partial\Omega$. We generalize the famous result of Brezis and Nirenberg [H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.* **317**(5) (1993) 465–472] to a more general class of functionals and extend all the other generalizations of this result which has been published in the last decades.

Keywords: Nonhomogeneous partial differential operator; local minimizer; Clarke's generalized gradient; critical growth; Neumann problem.

Mathematics Subject Classification: 35-XX

1. Introduction

Consider the following functional $\Phi: H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, s) = \int_0^s f(x, t) dt$ with a Carathéodory function $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ that satisfies the growth condition

$$|f(x, u)| \leq C(1 + |u|^p) \quad \text{with } p \leq \frac{N+2}{N-2}.$$

It is well known that a local $C_0^1(\overline{\Omega})$ -minimizer of Φ is also a local $H_0^1(\Omega)$ -minimizer of Φ . Such a result is originally due to Brezis and Nirenberg [3] for functionals on H_0^1 and the critical points of Φ are weak solutions of the equation

$$-\Delta u = f(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where Δ denotes the well-known Laplace differential operator. An extension of the result of Brezis and Nirenberg to functionals related with the p -Laplace differential operator was done by García Azorero *et al.* [6] who considered the functional $J_p: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, s) = \int_0^s f(x, t) dt$ and $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following growth condition:

$$|f(x, s)| \leq C(1 + |s|^{r-1}) \quad \text{with } r < \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

A simpler proof than those in [6] but only in case $p > 2$ was done by Guo and Zhang [11]. A nonsmooth version for functionals defined on $W_0^{1,p}(\Omega)$ with $p \geq 2$ has been studied by Motreanu and Papageorgiou [17].

The first paper concerning local minimizers of functional corresponding to non-linear parametric Neumann problems was written by Motreanu *et al.* [16]. Therein, the potential $\Phi_0: W_n^{1,p}(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\Phi_0(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z F_0(z, x(z)) dz, \quad 1 < p < \infty$$

with

$$W_n^{1,p}(\Omega) = \left\{ x \in W^{1,p}(\Omega) : \frac{\partial x}{\partial n} = 0 \right\},$$

where $\frac{\partial x}{\partial n}$ is the outer normal derivative of x and $F_0(z, x) = \int_0^x f_0(z, s) ds$. The first result dealing with nonsmooth functionals defined on $W_n^{1,p}(\Omega)$ for the case $2 \leq p < \infty$ was proved by Barletta and Papageorgiou [2] while the general case $1 < p < \infty$ has been treated by Iannizzotto and Papageorgiou [13]. The first result concerning functionals defined on $W^{1,p}(\Omega)$ involving a boundary term was published by the third author in the smooth [21] and in the nonsmooth [22] case. Moreover, a singular functional $I: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{p} \|u\|_{W_0^{1,p}(\Omega)}^p - \int_\Omega F(x, u^+) dx - \int_\Omega G(u^+) dx,$$

with $F(x, t) = \int_0^t f(x, s) ds$ and $G(t) = \int_0^t g(s) ds$ with $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ being a singular term such that $\lim_{t \rightarrow 0^+} g(t) = +\infty$ was studied by Giacomoni and Saoudi [10].

All the above-mentioned works are related to the p -Laplace differential operator. A first result concerning local minimizers and nonhomogeneous operators was presented in the work of Motreanu and Papageorgiou [18] who studied functionals of the form

$$\varphi_0(u) = \int_\Omega G(x, \nabla u) dx - \int_\Omega F_0(x, u) dx, \quad u \in W_n^{1,p}(\Omega),$$

where G is the potential of a general nonhomogeneous operator. A prototype of such operator is the (p, q) -Laplace differential operator which is the sum of the p - and q -Laplacian. A nonsmooth version of functionals related to nonhomogeneous operators defined on the space $W^{1,p}(\Omega)$ has been studied by Gasiński and Papageorgiou [8].

Recently, Papageorgiou and Rădulescu [19] studied functionals that are not only related to nonhomogeneous operator but also have a boundary term and the potential term in the domain is related to a Carathéodory function that has critical growth. Namely, they considered the functional $\varphi_0: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du) dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) |u|^p d\sigma - \int_\Omega F_0(z, u) dz,$$

where $F_0(z, x) = \int_0^x f_0(z, s) ds$ and $f_0(x, \cdot)$ has critical growth.

In this paper, we are interested in a generalization of all the above-mentioned results. The idea is to study functionals on $W^{1,p}(\Omega)$ which are related to

nonhomogeneous operators and involving boundary terms that allow critical growth also at the boundary.

To this end, let $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ be a bounded domain with a $C^{1,\alpha}$ -boundary $\partial\Omega$ and consider the following functional $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J(u) = \int_{\Omega} G(x, \nabla u) dx - \int_{\Omega} j_1(x, u) dx - \int_{\partial\Omega} j_2(x, u) d\sigma, \quad (1.1)$$

where $G(x, \cdot)$ is the primitive of a function $a(x, \cdot)$ and the nonlinearities $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable in the first argument and locally Lipschitz in the second one, that is, for every $s \in \mathbb{R}$ there exist a neighborhood $U_{s,k}$ of s and a constant $L_{s,k} \geq 0$ such that

$$|j_k(x, r) - j_k(x, t)| \leq L_{s,k} |r - t| \quad \text{for all } r, t \in U_{s,k}, \text{ for } k = 1, 2,$$

and for all $x \in \Omega$ and for all $x \in \partial\Omega$, respectively. It is easy to see that $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ need not to be differentiable and clearly it corresponds to the following elliptic inclusion:

$$\begin{aligned} -\operatorname{div} a(x, \nabla u) &\in \partial j_1(x, u) \quad \text{in } \Omega, \\ a(x, \nabla u) \cdot \nu &\in \partial j_2(x, \gamma u) \quad \text{on } \partial\Omega, \end{aligned}$$

where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$ and $\partial j_k(x, u)$, $k = 1, 2$, stands for Clarke's generalized gradient given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^\circ(x, s; r) \geq \xi r, \text{ for all } r \in \mathbb{R}\},$$

where the term $j_k^\circ(x, s; r)$ denotes the generalized directional derivative of the locally Lipschitz function $s \mapsto j_k(x, s)$ at s in the direction r defined by

$$j_k^\circ(x, s; r) = \limsup_{y \rightarrow s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t},$$

see [5, Chap. 2]. Based on the Hahn–Banach theorem, the set $\partial j_k(x, s)$ is nonempty. An element $u \in \mathbb{R}$ is said to be a critical point of a locally Lipschitz function $f: X \rightarrow \mathbb{R}$ if there holds

$$f^\circ(x; y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$ (see [4]).

2. Preliminaries and Hypotheses

For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ and $L^p(\Omega, \mathbb{R}^N)$ the standard Lebesgue spaces equipped with the norm $\|\cdot\|_p$ and, for $1 < p < \infty$, $W^{1,p}(\Omega)$ denotes the Sobolev spaces endowed with the norm $\|\cdot\|_{1,p}$. Duality pairing between $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)^*$ will be denoted by $\langle \cdot, \cdot \rangle$.

On the boundary $\partial\Omega$ we consider the $(N - 1)$ -dimensional Hausdorff (surface) measure σ . Having this measure, we can consider the boundary Lebesgue spaces $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$ with norm $\|\cdot\|_{q,\partial\Omega}$. Furthermore, we know that there exists

a unique linear, continuous map $\gamma: W^{1,p}(\Omega) \rightarrow L^q(\partial\Omega)$ for $1 \leq q \leq p_*$ called the trace map such that

$$\gamma(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}),$$

where p_* is the critical exponent on the boundary given by

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ \text{any } q \in (1, \infty) & \text{if } p \geq N. \end{cases} \quad (2.1)$$

Having the trace operator, we can talk about the boundary values for an arbitrary Sobolev function. Within the paper, we will omit the usage of the trace operator γ , for the sake of notational simplicity. Whenever considering the values of a Sobolev function on $\partial\Omega$, we understand that the trace operator is applied.

Furthermore, the Sobolev embedding theorem guarantees the existence of a linear, continuous map $i: W^{1,p}(\Omega) \rightarrow L^{p^*}(\Omega)$ with the critical exponent in the domain given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \text{any } q \in (1, \infty) & \text{if } p \geq N. \end{cases} \quad (2.2)$$

For more information on the Sobolev embeddings we refer to Gasiński and Papageorgiou [9] or Adams [1].

For $s \in (1, +\infty)$ we denote by $s' = \frac{s}{s-1}$ its conjugate, the inner product in \mathbb{R}^N is denoted by \cdot and the norm of \mathbb{R}^N is given by $|\cdot|$. Moreover, $\mathbb{R}_+ = [0, +\infty)$ and the Lebesgue measure is denoted by $|\cdot|_N$.

Next, let $\vartheta \in C^1(0, \infty)$ be any function satisfying

$$0 < a_1 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq a_2 \quad \text{and} \quad a_3 t^{p-1} \leq \vartheta(t) \leq a_4 (t^{q-1} + t^{p-1}) \quad (2.3)$$

for all $t > 0$, with some constants $a_i > 0$, $i \in \{1, 2, 3, 4\}$ and for $1 < q < p < \infty$. The hypotheses on $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ are listed as follows:

H(a): $a(x, \xi) = a_0(x, |\xi|)\xi$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$ for all $\xi \in \mathbb{R}^N$ and with $a_0(x, t) > 0$ for all $x \in \overline{\Omega}$, for all $t > 0$ and

- (i) $a_0 \in C^1(\overline{\Omega} \times (0, \infty))$, $t \mapsto ta_0(x, t)$ is strictly increasing in $(0, \infty)$,
 $\lim_{t \rightarrow 0^+} ta_0(x, t) = 0$ for all $x \in \overline{\Omega}$ and

$$\lim_{t \rightarrow 0^+} \frac{ta_0'(x, t)}{a_0(x, t)} = c > -1 \quad \text{for all } x \in \overline{\Omega};$$

- (ii) $|\nabla_\xi a(x, \xi)| \leq a_5 \frac{\vartheta(|\xi|)}{|\xi|}$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for some $a_5 > 0$;
 (iii) $\nabla_\xi a(x, \xi)y \cdot y \geq \frac{\vartheta(|\xi|)}{|\xi|}|y|^2$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for all $y \in \mathbb{R}^N$.

Remark 2.1. The idea in the choice of the special structure in $H(a)$ is the usage of the nonlinear regularity theory due to Lieberman [14] coupled with the nonlinear maximum principle of Pucci and Serrin [20] as well as Zhang [23] when considering certain differential equations. If we set

$$G_0(x, t) = \int_0^t a_0(x, s) s ds,$$

then $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ and the function $G_0(x, \cdot)$ is increasing and strictly convex for all $x \in \overline{\Omega}$. We set $G(x, \xi) = G_0(x, |\xi|)$ for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^N$ and obtain that $G \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and that the function $\xi \rightarrow G(x, \xi)$ is convex. Moreover, we easily derive that

$$\nabla_\xi G(x, \xi) = (G_0)'_t(x, |\xi|) \frac{\xi}{|\xi|} = a_0(x, |\xi|) \xi = a(x, \xi)$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\nabla_\xi G(x, 0) = 0$. In other words, $G(x, \cdot)$ occurs to be the primitive of $a(x, \cdot)$. Combining this with convexity of $G(x, \cdot)$ and the fact that $G(x, 0) = 0$ for all $x \in \overline{\Omega}$ we get

$$G(x, \xi) \leq a(x, \xi) \cdot \xi \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N. \quad (2.4)$$

The following lemma summarizes some properties of the function $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$.

Lemma 2.2. *If hypotheses $H(a)$ hold, then:*

- (i) $a \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$ and for all $x \in \overline{\Omega}$ the map $\xi \mapsto a(x, \xi)$ is continuous, strictly monotone and so maximal monotone as well;
- (ii) there exists $a_6 > 0$, such that $|a(x, \xi)| \leq a_6 (1 + |\xi|^{p-1})$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$;
- (iii) $a(x, \xi) \cdot \xi \geq \frac{a_3}{p-1} |\xi|^p$ for all $x \in \overline{\Omega}$ and for all $\xi \in \mathbb{R}^N$.

Lemma 2.2 together with (2.4) allow to obtain the following growth estimates on $G(x, \cdot)$.

Corollary 2.3. *If hypotheses $H(a)$ hold, then there exists $a_7 > 0$ such that*

$$\frac{a_3}{p(p-1)} |\xi|^p \leq G(x, \xi) \leq a_7 (1 + |\xi|^p)$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

The nonlinear operator $A: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx \quad \text{for all } u, \varphi \in W^{1,p}(\Omega), \quad (2.5)$$

possesses the following useful properties (see Gasiński and Papageorgiou [9]).

Proposition 2.4. *If hypotheses $H(a)$ hold and the operator $A: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is defined by (2.5), then A is bounded, monotone, continuous, hence maximal monotone and of type (S_+) .*

The following examples expose some operators fitting in our setting.

Example 2.5. In the definitions of the operators a , we drop the dependence on x just for simplicity. All the following maps satisfy hypotheses $H(a)$:

- (i) If $a(\xi) = |\xi|^{p-2}\xi$ with $1 < p < \infty$, then the corresponding operator is the classical p -Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

In this case $G(\xi) = \frac{1}{p}|\xi|^p$ for all $\xi \in \mathbb{R}^N$.

- (ii) If $a(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$ with $1 < q < p < \infty$ and $\mu > 0$ then the corresponding operator is the so-called weighted (p, q) -Laplacian defined by $\Delta_p u + \mu \Delta_q u$ for all $u \in W^{1,p}(\Omega)$. In this case $G(\xi) = \frac{1}{p}|\xi|^p + \frac{\mu}{q}|\xi|^q$ for all $\xi \in \mathbb{R}^N$.
- (iii) If $a(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}}\xi$ with $1 < p < \infty$, then this map represents the generalized p -mean curvature differential operator defined by

$$\operatorname{div}[(1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] \quad \text{for all } u \in W^{1,p}(\Omega).$$

In this case $G(\xi) = \frac{1}{p}(1 + |\xi|^2)^{\frac{p}{2}}$ for all $\xi \in \mathbb{R}^N$.

Next, let us give the hypotheses on the nonsmooth potentials $j_1: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $j_2: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$.

- $H(j_1)$ (i) $x \mapsto j_1(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$;
(ii) $s \mapsto j_1(x, s)$ is locally Lipschitz for almost all $x \in \Omega$;
(iii) for some constants $c_1 > 0$ and $1 < q_1 \leq p^*$ (where p^* is the given in (2.2)), we have

$$|\xi_1| \leq c_1(1 + |s|^{q_1-1})$$

for almost all $x \in \Omega$ and for all $\xi_1 \in \partial j_1(x, s)$.

- $H(j_2)$ (i) $x \mapsto j_2(x, s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$;
(ii) $s \mapsto j_2(x, s)$ is locally Lipschitz for almost all $x \in \partial\Omega$;
(iii) for some constants $c_2 > 0$ and $1 < q_2 \leq p_*$ (where p_* is given in (2.1)), we have

$$|\xi_2| \leq c_2(1 + |s|^{q_2-1})$$

for almost all $x \in \partial\Omega$ and all $\xi_2 \in \partial j_2(x, s)$;

- (iv) for any $u \in W^{1,p}(\Omega)$ and $\xi_3 \in \partial j_2(x, u)$ we have

$$|\xi_3(x_1) - \xi_3(x_2)| \leq L|x_1 - x_2|^\alpha,$$

for all x_1, x_2 in $\partial\Omega$ with $\alpha \in (0, 1]$.

3. Main Result

The following main result of this paper gives an answer about the relation between local Sobolev and Hölder minimizers of functionals of type J given in (1.1). We point out again that our functional is more general than the functionals of all the other cited papers above because we have a general, nonhomogeneous operator and we allow critical growth even on the boundary.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ be a bounded domain with a $C^{1,\alpha}$ -boundary $\partial\Omega$ and let the assumptions $H(a)$, $H(j_1)$, and $H(j_2)$ be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$ -minimizer of J , that is, there exists $\rho_0 > 0$ such that*

$$J(u_0) \leq J(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$ and u_0 is a local $W^{1,p}(\Omega)$ -minimizer of J , that is, there exists $\rho_1 > 0$ such that

$$J(u_0) \leq J(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

Proof. First, from hypotheses $H(a)$, $H(j_1)$, $H(j_2)$ and Hu and Papageorgiou [12, p. 313], we know that the functional $J: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ is locally Lipschitz continuous. Let $h \in C^1(\overline{\Omega})$ and let $t > 0$ be small. Since u_0 is a local $C^1(\overline{\Omega})$ -minimizer of J , we have

$$0 \leq \frac{J(u_0 + th) - J(u_0)}{t}.$$

This implies

$$0 \leq J^\circ(u_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$

Note that the function $h \mapsto J^\circ(u_0; h)$ is upper semicontinuous and $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, hence

$$0 \leq J^\circ(u_0; h) \quad \text{for all } h \in W^{1,p}(\Omega).$$

Obviously, we have

$$0 \in \partial J(u_0).$$

This means that there exist functions $g_1 \in L^{q'_1}(\Omega)$ with $g_1(x) \in \partial j_1(x, u_0(x))$ for almost all $x \in \Omega$ and $g_2 \in L^{q'_2}(\partial\Omega)$ with $g_2(x) \in \partial j_2(x, u_0(x))$ for almost all $x \in \partial\Omega$ such that

$$\int_{\Omega} a(x, \nabla u_0) \cdot \nabla v dx = \int_{\Omega} g_1 v dx + \int_{\partial\Omega} g_2 v d\sigma \quad \text{for all } v \in W^{1,p}(\Omega). \quad (3.1)$$

Equation (3.1) stands for the weak formulation of the following nonhomogeneous Neumann boundary value problem:

$$-\operatorname{div} a(x, \nabla u_0) = g_1 \quad \text{in } \Omega, \quad a(x, \nabla u_0) \cdot \nu = g_2 \quad \text{on } \partial\Omega.$$

It follows from Marino and Winkert [15, Theorem 3.1] that $u_0 \in L^\infty(\overline{\Omega})$. This combined with the regularity results due to Lieberman [14] implies the existence of

$\eta \in (0, 1)$ and $M > 0$ such that

$$u_0 \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_0\|_{C^{1,\eta}(\overline{\Omega})} \leq M. \quad (3.2)$$

To obtain our thesis, we need to show that u_0 is also a local minimizer of J in the $W^{1,p}(\Omega)$ -norm. For this purpose, consider the minimizing problem

$$m_0^\varepsilon = \inf_{h \in \overline{B}_\varepsilon} J(u_0 + h), \quad (3.3)$$

where

$$\overline{B}_\varepsilon = \{h \in W^{1,p}(\Omega) \mid \|h\|_{1,p} \leq \varepsilon\}.$$

Arguing by contradiction, assume that u_0 is not a local minimizer of the functional J in the $W^{1,p}(\Omega)$ -topology. Then we find $\varepsilon_0 \in (0, 1]$ such that

$$m_0^\varepsilon < J(u_0) \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \quad (3.4)$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\{h_n\}_{n \geq 1} \subset \overline{B}_\varepsilon$ be a minimizing sequence for (3.3), that is

$$\lim_{n \rightarrow \infty} J(u_0 + h_n) = m_0^\varepsilon. \quad (3.5)$$

From (3.4), we see that $\|\nabla h_n\|_p$ is bounded and since $u \mapsto \|\nabla u\|_p + \|u\|_{p^*}$ is an equivalent norm on $W^{1,p}(\Omega)$ (we can also use the norm $u \mapsto \|\nabla u\|_p + \|u\|_{p^*, \partial\Omega}$), it is clear that the sequence $\{h_n\}_{n \geq 1} \subseteq \overline{B}_\varepsilon$ is bounded in $W^{1,p}(\Omega)$ and so we can assume that

$$h_n \rightharpoonup h_\varepsilon \quad \text{in } W^{1,p}(\Omega), \quad \text{in } L^{p^*}(\Omega) \quad \text{and in } L^{p^*}(\partial\Omega), \quad (3.6)$$

$$h_n(x) \rightarrow h_\varepsilon(x) \quad \text{for almost all } x \in \Omega \quad \text{and for almost all } x \in \partial\Omega,$$

by the Sobolev and the trace embedding theorem, respectively.

Applying the Extended Fatou Lemma (see, [7, Theorem A.2.8]), we can obtain that φ is sequentially weakly semicontinuous. From (3.5) and (3.6) it follows that

$$m_0^\varepsilon = \inf_{h \in \overline{B}_\varepsilon} J(u_0 + h) \leq J(u_0 + h_\varepsilon) \leq \liminf_{n \rightarrow \infty} J(u_0 + h_n) \leq \lim_{n \rightarrow \infty} J(u_0 + h_n) = m_0^\varepsilon,$$

and hence, due to (3.4), $h_\varepsilon \neq 0$.

We are now in the position to apply the nonsmooth Lagrange multiplier rule, see [5, Theorem 1 and Proposition 13], which guarantees the existence of a multiplier $\lambda_\varepsilon \geq 0$ such that

$$0 \in \partial J(u_0 + h_\varepsilon) + \lambda_\varepsilon K(h_\varepsilon),$$

where the function $K: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is defined by

$$\langle K(h_\varepsilon), v \rangle = \int_\Omega |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v \, dx + \int_\Omega |h_\varepsilon|^{p-2} h_\varepsilon v \, dx \quad \text{for all } v \in W^{1,p}(\Omega).$$

Therefore,

there exist $\hat{g}_1 \in L^{q'_1}(\Omega)$ and $\hat{g}_2 \in L^{q'_2}(\partial\Omega)$ with $\hat{g}_1(x) \in \partial j_1(x, (u_0 + h_\varepsilon)(x))$ for almost all $x \in \Omega$ and $\hat{g}_2(x) \in \partial j_2(x, (u_0 + h_\varepsilon)(x))$ for almost all $x \in \partial\Omega$ such

that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla(u_0 + h_{\varepsilon})) \cdot \nabla v dx - \int_{\Omega} \hat{g}_1 v dx - \int_{\partial\Omega} \hat{g}_2 v d\sigma \\ & + \lambda_{\varepsilon} \int_{\Omega} |h_{\varepsilon}|^{p-2} h_{\varepsilon} v dx + \lambda_{\varepsilon} \int_{\Omega} |\nabla h_{\varepsilon}|^{p-2} \nabla h_{\varepsilon} \cdot \nabla v dx = 0 \end{aligned} \quad (3.7)$$

for all $v \in W^{1,p}(\Omega)$. We need to prove that h_{ε} belongs to $L^{\infty}(\Omega)$ and hence to $C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$ due to the regularity results due to Lieberman [14]. To end this, let us consider three cases for the multiplier λ_{ε} .

Case 1. $\lambda_{\varepsilon} = 0$ with $\varepsilon \in (0, 1]$

In this case, Eq. (3.7) becomes

$$\int_{\Omega} a(x, \nabla(u_0 + h_{\varepsilon})) \cdot \nabla v dx = \int_{\Omega} \hat{g}_1 v dx + \int_{\partial\Omega} \hat{g}_2 v d\sigma \quad \text{for all } v \in W^{1,p}(\Omega).$$

As before, by applying the *a priori* results of Marino and Winkert [15, Theorem 3.1], the regularity results due to Lieberman [14, Theorem 2] and the fact that $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$ gives

$$h_{\varepsilon} \in C^{1,\hat{\eta}}(\overline{\Omega}) \quad \text{and} \quad \|h_{\varepsilon}\|_{C^{1,\hat{\eta}}(\overline{\Omega})} \leq M \quad (3.8)$$

for some $\hat{\eta} \in (0, 1)$ and $M > 0$.

Case 2. $0 < \lambda_{\varepsilon} \leq 1$ with $\varepsilon \in (0, 1]$

Multiplying (3.1) by $\lambda_{\varepsilon} > 0$ and adding this to (3.7) results in

$$\begin{aligned} & \int_{\Omega} a(x, \nabla(u_0 + h_{\varepsilon})) \cdot \nabla v dx + \lambda_{\varepsilon} \int_{\Omega} a(x, \nabla u_0) \cdot \nabla v dx \\ & + \lambda_{\varepsilon} \int_{\Omega} |\nabla h_{\varepsilon}|^{p-2} \nabla h_{\varepsilon} \cdot \nabla v dx \\ & = \int_{\Omega} (-\lambda_{\varepsilon} |h_{\varepsilon}|^{p-2} h_{\varepsilon} + \hat{g}_1 + \lambda_{\varepsilon} g_1) v dx + \int_{\partial\Omega} (\hat{g}_2 + \lambda_{\varepsilon} g_2) v d\sigma. \end{aligned} \quad (3.9)$$

Now we introduce the map $T_{\varepsilon}: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by

$$T_{\varepsilon}(x, \xi) = a(x, \xi) + \lambda_{\varepsilon} a(x, H(x)) + \lambda_{\varepsilon} |\xi - H(x)|^{p-2} (\xi - H(x))$$

for all $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$, where $H(x) = \nabla u_0(x)$ and $H \in C^{\eta}(\overline{\Omega}; \mathbb{R}^N)$ for some $\eta \in (0, 1)$, thanks to (3.2). Since $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous (see Lemma 2.2(i)), let $m_H = \max_{x \in \overline{\Omega}} |a(x, H(x))| = \max_{x \in \overline{\Omega}} |a(x, \nabla u_0(x))|$. It is easy to see that $T_{\varepsilon} \in C(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^N)$. On the other side, we can apply Lemma 2.2(iii) and Young's inequality to obtain

$$\begin{aligned} T_{\varepsilon}(x, \xi) \cdot \xi &= a(x, \xi) \cdot \xi + \lambda_{\varepsilon} a(x, H(x)) \cdot \xi + \lambda_{\varepsilon} |\xi - H(x)|^{p-2} (\xi - H(x)) \cdot \xi \\ &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_{\varepsilon} |a(x, H(x))| \cdot |\xi| + \lambda_{\varepsilon} |\xi - H(x)|^p \\ &\quad - \lambda_{\varepsilon} |\xi - H(x)|^{p-2} (\xi - H(x)) \cdot H(x) \end{aligned}$$

$$\begin{aligned} &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_\varepsilon m_H |\xi| - \lambda_\varepsilon |\xi - H(x)|^{p-1} |H(x)| \\ &\geq \frac{a_3}{p-1} |\xi|^p - \lambda_\varepsilon m_H |\xi| - \lambda_\varepsilon m_H |\xi - H(x)|^{p-1} \\ &\geq \frac{a_3}{p-1} |\xi|^p - \delta |\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta), \end{aligned}$$

where $\delta = \frac{a_3}{2(p-1)}$ and $d_1(\lambda_\varepsilon, m_H, \delta) > 0$ is a constant, which is independent of ξ . Hence, we have

$$T_\varepsilon(x, \xi) \cdot \xi \geq \frac{a_3}{2(p-1)} |\xi|^p - d_1(\lambda_\varepsilon, m_H, \delta)$$

for all $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$. This means that T_ε satisfies a strong ellipticity condition. Note that Eq. (3.9) can be written in the form

$$\begin{aligned} -\operatorname{div}(T_\varepsilon(x, \nabla(u_0 + h_\varepsilon))) &= -\lambda_\varepsilon |h_\varepsilon|^{p-2} h_\varepsilon + \hat{g}_1 + \lambda_\varepsilon g_1 \quad \text{in } \Omega, \\ T_\varepsilon(x, \nabla(u_0 + h_\varepsilon)) \cdot \nu &= \hat{g}_2 + \lambda_\varepsilon g_2 \quad \text{on } \partial\Omega. \end{aligned} \quad (3.10)$$

Now are able to apply the again the results of Marino and Winkert [15, Theorem 3.1] which gives $u_0 + h_\varepsilon \in L^\infty(\bar{\Omega})$. However, $u_0 \in C^{1,\eta}(\bar{\Omega})$ leads to $h_\varepsilon \in L^\infty(\bar{\Omega})$. Moreover, by using (2.3) and hypothesis H(a)(ii), we obtain

$$\begin{aligned} |\nabla_\xi T_\varepsilon(x, \xi)| &\leq |\nabla_\xi a(x, \xi)| + \lambda_\varepsilon |\nabla_\xi [|\xi - H(x)|^{p-2} (\xi - H(x))]| \\ &\leq a_5 \frac{\vartheta(|\xi|)}{|\xi|} + b_1 + b_2 |\xi|^{p-2} \\ &\leq a_5 a_4 (1 + 2|\xi|^{p-2}) + b_1 + b_2 |\xi|^{p-2} \\ &= (2a_4 a_5 + b_2) |\xi|^{p-2} + b_1 + a_4 a_5 \end{aligned} \quad (3.11)$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$, for almost all $x \in \Omega$ and for some $b_1, b_2 > 0$ which are independent of ξ . In the same way, applying (2.3) and hypothesis H(a)(iii) leads to

$$\begin{aligned} \nabla_\xi T_\varepsilon(x, \xi) y \cdot y &= \nabla_\xi a(x, \xi) y \cdot y + \lambda_\varepsilon \nabla_\xi [|\xi - H(x)|^{p-2} (\xi - H(x))] y \cdot y \\ &\geq \frac{\vartheta(|\xi|)}{|\xi|} |y|^2 + \lambda_\varepsilon |\xi - H(x)|^{p-2} |y|^2 \\ &\quad + \lambda_\varepsilon (p-2) |\xi - H(x)|^{p-4} (\xi - H(x)) \cdot y \\ &\geq c_1 |\xi|^{p-2} |y|^2 + \lambda_\varepsilon \min\{1, p-1\} |\xi - H(x)|^{p-2} |y|^2 \\ &\geq c_1 |\xi|^{p-2} |y|^2. \end{aligned} \quad (3.12)$$

Finally, since $h_\varepsilon \in L^\infty(\Omega)$ satisfies (3.10) and because of H(a), (3.11), (3.12) along with hypotheses H(j₁) and H(j₂) we are able to apply the regularity results due to Lieberman [14] which gives (3.8) in Case 2 as well.

Case 3. $\lambda_\varepsilon > 1$ with $\varepsilon \in (0, 1]$

Multiplying (3.1) by -1 and adding this to (3.7) results in

$$\begin{aligned} & \int_{\Omega} a(x, \nabla(u_0 + h_\varepsilon)) \cdot \nabla v dx - \int_{\Omega} a(x, \nabla u_0) \cdot \nabla v dx + \lambda_\varepsilon \int_{\Omega} |\nabla h_\varepsilon|^{p-2} \nabla h_\varepsilon \cdot \nabla v dx \\ &= \int_{\Omega} (\hat{g}_1 - g_1 - \lambda_\varepsilon |h_\varepsilon|^{p-2} h_\varepsilon) v(x) dx + \int_{\partial\Omega} (\hat{g}_2 - g_2) d\sigma. \end{aligned} \quad (3.13)$$

As before, we define a map $T_\varepsilon: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$T_\varepsilon(x, \xi) = \frac{1}{\lambda_\varepsilon} (a(x, H(x) + \xi) - a(x, H(x))) + |\xi|^{p-2} \xi$$

for all $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$, where $H(x) = \nabla u_0(x)$ with $H \in C^\eta(\overline{\Omega}; \mathbb{R}^N)$ for some $\eta \in (0, 1)$ because of (3.2). Applying the notation for T_ε we can rewrite (3.13) in the following sense:

$$\begin{aligned} -\operatorname{div}(T_\varepsilon(x, \nabla h_\varepsilon)) &= \frac{1}{\lambda_\varepsilon} (\hat{g}_1 - g_1) - |h_\varepsilon|^{p-2} h_\varepsilon \quad \text{in } \Omega, \\ T_\varepsilon(x, \nabla h_\varepsilon) \cdot \nu &= \frac{1}{\lambda_\varepsilon} (\hat{g}_2 - g_2) \quad \text{on } \partial\Omega. \end{aligned}$$

As before we can easily show that

$$\begin{aligned} \nabla_\xi T_\varepsilon(x, \xi) y \cdot y &\geq b_3 |\xi|^{p-2} |y|^2, \\ T_\varepsilon(x, \xi) \cdot \xi &\geq b_4 |\xi|^p + b_5, \\ |\nabla_\xi T_\varepsilon(x, \xi)| &\leq b_6 |\xi|^{p-2} + b_7, \end{aligned}$$

for some positive constants b_3, b_4, b_5, b_6, b_7 . Finally, applying Marino and Winkert [15, Theorem 3.1] and Lieberman [14, Theorem 2] we reach again (3.8) in Case 3.

Let $\varepsilon \downarrow 0$. By the compactness of the embedding $C^{1, \hat{\eta}}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ (see [1, p.11]), there exists a subsequence $\{h_{\varepsilon_n}\}_{n \geq 1}$ of $\{h_\varepsilon\}$ and a function $h^* \in C^1(\overline{\Omega})$ such that

$$h_{\varepsilon_n} \rightarrow h^* \quad \text{in } C^1(\overline{\Omega}).$$

Note that $h_{\varepsilon_n} \in \overline{B}_{\varepsilon_n}$ which gives $h^* = 0$. Therefore, we are able to find $N_0 \in \mathbb{N}$ large enough such that

$$\|h_{\varepsilon_n}\|_{C^1(\overline{\Omega})} \leq r_1 \quad \text{for all } n \geq N_0.$$

Because u_0 is a minimizer of J in the $C^1(\overline{\Omega})$ -topology, we have

$$J(u_0) \leq J(u_0 + h_{\varepsilon_n}).$$

However, by the choice of $\{h_{\varepsilon_n}\}_{n \geq 1}$, it holds

$$J(u_0 + h_{\varepsilon_n}) = m_{\varepsilon_n}^0 < J(u_0).$$

which is a contradiction. Therefore, we conclude that u_0 is a local minimizer of J in the $W^{1,p}(\Omega)$ -topology. \square

Let us comment on the case where the functional is smooth. Let $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be Carathéodory functions, that means, we assume measurability in the first argument and continuity in the second one. We define $F(x, s) = \int_0^s f(x, t)dt$, $H(x, s) = \int_t^s h(x, t)dt$ and consider the functional $I: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$I(u) = \int_{\Omega} G(x, \nabla u)dx - \int_{\Omega} F(x, u)dx - \int_{\partial\Omega} H(x, u)d\sigma. \quad (3.14)$$

Of course, $I \in C^1(W^{1,p}(\Omega))$. For the functions $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ we suppose the existence of constants $c_1, c_2 > 0$ such that

$$\begin{aligned} |f(x, s)| &\leq c_1(1 + |s|^{q_1-1}) \quad \text{for almost all } x \in \Omega, \\ |h(x, s)| &\leq c_2(1 + |s|^{q_2-1}) \quad \text{for almost all } x \in \partial\Omega, \end{aligned} \quad (3.15)$$

for all $s \in \mathbb{R}$ and for $1 < q_1 \leq p^*$ as well as $1 < q_2 \leq p_*$. Moreover, $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition

$$|h(x, s) - h(y, t)| \leq L[|x - y|^\alpha + |s - t|^\alpha], \quad |g(x, s)| \leq L \quad (3.16)$$

for all $(x, s), (y, t) \in \partial\Omega \times [-M_0, M_0]$ with $\alpha \in (0, 1]$ and constants $M_0 > 0$ and $L \geq 0$.

Then, Theorem 3.1 states the following for the functional $I: W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined in (3.14).

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ be a bounded domain with a $C^{1,\alpha}$ -boundary $\partial\Omega$ and let the assumptions $H(a)$, (3.15) and (3.16) be satisfied. If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$ -minimizer of I , that is, there exists $\rho_0 > 0$ such that*

$$I(u_0) \leq I(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0, 1)$ and u_0 is a local $W^{1,p}(\Omega)$ -minimizer of I , that is, there exists $\rho_1 > 0$ such that

$$I(u_0) \leq I(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

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